Transient conduction in a plate cooled by free convection

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An approximate analytical solution of the one-dimensional conduction equation with a natural convection boundary condition is presented. The solution is based on the heat balance integral technique and possesses considerable utility. The accuracy of the solution is tested by comparison with an exact solution for a range of linear forced convection problems and with a Crank–Nicolson solution for a range of nonlinear free convection problems. It is demonstrated that significant differences can occur between the temperature responses of a solid cooled by either free or forced convective flow at similar Biot numbers.

Keywords: *transient conduction, free convection*

For heat transfer processes where the surface heat transfer mechanism is nonlinear in temperature no analytical solutions exist. An approximation to such problems may sometimes be made by linearization of the surface boundary condition. This approximation is often used to obtain solutions for transient conduction problems with a natural convection boundary condition where the temperature dependence of the convection coefficient is ignored. The surrogate problem involving pseudo-forced convection heat transfer at the surface is then readily solved analytically. Under certain circumstances this approach can be unsatisfactory. The purpose of this paper is to demonstrate that an approximate analytical solution for the free convection cooling (or heating) of a plate does exist, that the solution is reasonably accurate for many purposes and that it is easy to use. The approximate solution is derived using the heat balance integral technique, the merits of which have been discussed by Goodman[1].

Statement of problem

It is required to predict the transient temperature distribution in a plate which is initially at a uniform temperature T_i throughout. At time $t=0$ one face of the plate ($x=0$) is suddenly subject to cooling by free convection. The other face of the plate ($x=L$) is adiabatic. The problem is mathematically similar to that of a plate of thickness $2L$ which is subjected to simultaneous and identical heat transfer at both faces.

The assumptions are as follows.

(1) The heat conduction in the plate is one-dimensional.
(2) The solid is homogeneous, isotropic and the physical properties are independent of temperature.
(3) The ambient fluid temperature T_a is constant.

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In terms of dimensionless parameters the problem may be posed as follows.

$$\frac{\partial\theta}{\partial F}=\frac{\partial^2\theta}{\partial X^2} \quad 0\leqslant X\leqslant 1, \quad F>0 \tag{1}$$

$$\theta(X,0)=1 \tag{2}$$

$$\frac{\partial\theta}{\partial X}(1,F)=0 \tag{3}$$

$$\frac{\partial\theta}{\partial X}(0,F)=B(\eta-U)^n \quad 1\leqslant n\leqslant\frac{4}{3} \tag{4}$$

where the Biot number B is defined in terms of the heat transfer coefficient $\bar{h}=C_0(T_s-T_a)^{n-1}$, in which C_0 is a constant.

When $n=1$ the problem reduces to the linear case of forced convection. When $n=4/3$ the problem is that of an upward-facing surface cooling under the influence of a turbulent natural convection flow.

When the conditions at the solid boundary are such that laminar natural convection flow occurs, then a local variation in the value of $\bar{h}$ is likely. Under certain circumstances, such as with low Prandtl number fluids, it may be acceptable to assume a spatially constant value for $\bar{h}$, in which case the appropriate value for n would be 5/4.

The components of the integral solution

For transient conduction in the plate the integral solution of the energy equation for conduction in a semi-infinite body may be used up to the time at which the conduction front propagating from the heat transfer boundary reaches the adiabatic face. This is referred to as the penetration time. In the post-penetration period a complementary solution of the conduction equation must be used which satisfies the new boundary conditions prevailing in this period. These two components of the solution are derived below and are matched at the penetration time.

Transient conduction in a semi-infinite solid

The heat balance integral technique is conceptually similar to the integral method of boundary layer analysis. For a semi-infinite solid, initially at uniform temperature throughout, a temperature disturbance at the free boundary propagates in the x-direction through the solid. The leading edge of the thermal wave is defined to be at $x=\delta$ at time t from the initiation of the wavefront. The descent from a partial differential equation (Eq (1)) through an ordinary differential equation to an algebraic expression and finally to a numerical result is parallel to the process used in boundary layer analysis.

Eq (1) is integrated over the region occupied by the thermal wave, ie $0 \leqslant X \leqslant \delta^*$, giving:

$$\int_0^{\delta^*} \frac{\partial \theta}{\partial F} \mathrm{d}X = \left[\frac{\partial \theta}{\partial X}\right]_{X=\delta^*} - \left[\frac{\partial \theta}{\partial X}\right]_{X=0} \quad (5)$$

Eq (5) may be transformed into an ordinary differential equation for $\eta(F)$ as follows. First, the left hand side of Eq (5) may be rewritten using the general relationship:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,y)\,\mathrm{d}y = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f\,\mathrm{d}y + f(b,x)\frac{\mathrm{d}b}{\mathrm{d}x} - f(a,x)\frac{\mathrm{d}a}{\mathrm{d}x}$$

Eq (5) thus becomes:

$$\frac{\mathrm{d}}{\mathrm{d}F}\int_0^{X=\delta^*} \theta\,\mathrm{d}X - \theta_{X=\delta^*}\frac{\mathrm{d}\delta^*}{\mathrm{d}F} = \left[\frac{\partial \theta}{\partial X}\right]_{X=\delta^*} - \left[\frac{\partial \theta}{\partial X}\right]_{X=0} \quad (6)$$

The next step involves the substitution of an approximating polynomial for $\theta(F,X)$.

Several approximations for the temperature distribution are possible and have been used in previous applications of the heat balance integral technique, but Langford[2] has shown that a quadratic is usually adequate for plane flows.

For the present purposes the temperature distribution across the thermal wave is assumed to be of the form:

$$\theta = a_0 + a_1 X + a_2 X^2 \quad \text{where} \quad a_i(F),\ i=0,1,2 \quad (7)$$

The boundary conditions on this profile are:

at $X=\delta^*$, $\theta=1$, $\dfrac{\partial \theta}{\partial X}=0$

at $X=0$, $\theta=\eta$

These boundary conditions together with Eq (7) give:

$$a_0 = \eta \quad (8a)$$

$$a_2 = \frac{-a_1^2}{4(1-\eta)} \quad (8b)$$

$$\delta^* = \frac{2(1-\eta)}{a_1} \quad (8c)$$

Substitution of Eq (7) into Eq (6) gives:

$$\frac{\mathrm{d}}{\mathrm{d}F}\left[a_0\delta^* + a_1\frac{\delta^{*2}}{2} + a_2\frac{\delta^{*3}}{3}\right] - \left[a_0 + a_1\delta^* + a_2\delta^{*2}\right]\frac{\mathrm{d}\delta^*}{\mathrm{d}F} = \left[\frac{\partial \theta}{\partial X}\right]_{X=\delta^*} - \left[\frac{\partial \theta}{\partial X}\right]_{X=0} \quad (9)$$

Using Eqs (8) and noting that:

$$\left[\frac{\partial \theta}{\partial X}\right]_{X=\delta^*} = 0 \quad \text{and} \quad \left[\frac{\partial \theta}{\partial X}\right]_{X=0} = a_1$$

enables Eq (9) to be rewritten as:

$$\frac{2(1-\eta)}{a_1}\frac{\mathrm{d}\eta}{\mathrm{d}F} + \frac{2(1-\eta)^2}{a_1^2}\frac{\mathrm{d}a_1}{\mathrm{d}F} + \frac{8}{3}\frac{(1-\eta)^3}{a_1^3}\frac{\mathrm{d}a_2}{\mathrm{d}F} = -a_1 \quad (10)$$

Differentiation of Eq (8b) and substitution into Eq (10) gives:

$$\frac{4}{3}\frac{(1-\eta)}{a_1^2}\frac{\mathrm{d}\eta}{\mathrm{d}F} + \frac{2}{3}\frac{(1-\eta)^2}{a_1^3}\frac{\mathrm{d}a_1}{\mathrm{d}F} + 1 = 0 \quad (11)$$

But:

$$\frac{(1-\eta)^2}{a_1^3}\frac{\mathrm{d}a_1}{\mathrm{d}F} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}F}\left[\frac{(1-\eta)^2}{a_1^2}\right] - \frac{(1-\eta)}{a_1^2}\frac{\mathrm{d}\eta}{\mathrm{d}F}$$

and so Eq (11) may be rewritten as:

$$\frac{2}{3}\int_1^{\eta} \frac{(1-\eta)}{a_1^2}\mathrm{d}\eta - \frac{1}{3}\left[\frac{(1-\eta)}{a_1}\right]^2 = -\int_0^F \mathrm{d}F \quad (12)$$

Notation

- B Biot number, $\bar{h}L/k$
- C_0 A constant
- F Fourier number, $\alpha t/L^2$
- F^* Fourier number at the penetration time t^*
- $\bar{h}$ Heat transfer coefficient, $C_0(T_s - T_a)^{n-1}$
- k Thermal conductivity of solid
- L Plate thickness
- n Exponent in convection flux function
- t Time elapsed from start of cooling
- t^* Time at which conduction front reaches adiabatic face
- T Temperature in solid at position x
- T_a Ambient fluid temperature
- T_i Initial uniform temperature of plate
- T_s Surface temperature at time t
- U Dimensionless ambient temperature, T_a/T_i
- x Distance from cooled surface
- X Dimensionless distance, x/L
- α Thermal diffusivity of solid
- β Dimensionless surface temperature, T_s/T_a
- γ Dimensionless initial temperature, T_i/T_a
- δ Position of conduction front
- δ^* Dimensionless conduction front position, δ/L
- η Dimensionless surface temperature, T_s/T_i
- θ General dimensionless solid temperature, T/T_i
- λ_i Eigenvalues

which is a general and useful form of the heat balance integral for a semi-infinite solid in which the surface heat transfer mechanism is embodied in a_1.

For the particular case of convective cooling:

$$a_1 = B(\eta - U)^n \tag{13}$$

The integral in Eq (12) with a_1 given by Eq (13) may be easily evaluated, resulting in the final form of the heat balance for a convectively cooled semi-infinite solid:

$$B^2F = \frac{1}{3}\left[\frac{1-\eta}{(\eta-U)^n}\right]^2 + \frac{2}{3}\left[\frac{(1-U)}{(2n-1)(\eta-U)^{2n-1}} - \frac{1}{(2n-2)(\eta-U)^{2n-2}} + \frac{1}{(1-U)^{2n-2}}\left[\frac{1}{2n-2} - \frac{1}{2n-1}\right]\right] \tag{14}$$

This is an inverse solution from which the dimensionless time, B^2F, may be calculated for any chosen value of the dimensionless surface temperature η. The complete temperature profile is then available through Eqs (7) and (8).

Estimation of the penetration time

When the conduction front reaches the adiabatic face of the solid ($X=1$) the use of Eq (14) must be discontinued and a new solution used which takes into account the fact that the wave is no longer propagating. Instead the temperature at $X=1$ is changing continuously. Assuming that the semi-infinite solid solution applies up to and including the instant of penetration, F^*, then Eq (14) may be used to estimate F^* as follows:

The boundary conditions at penetration give:

$$a_1 = 2(1-\eta^*)$$

where η^* is the dimensionless surface temperature at penetration.

$$\therefore \quad B(\eta^* - U)^n + 2(\eta^* - 1) = 0 \tag{15}$$

which may be solved for η^* by the Newton–Raphson method. This value of η^* is then substituted into Eq (14) to give the corresponding value of F^* and hence sets the limit on the use of Eq (14).

The post-penetration solution

At times greater than F^* the heat balance integral is simplified, since $d\delta^*/dF = 0$ and $\delta^* = 1$.

During the post-penetration period the temperature profile adopts a new form, again assumed to be quadratic:

$$\theta = b_0 + b_1X + b_2X^2, \quad b_i(F), \quad i=0,1,2 \tag{16}$$

The new boundary conditions are that:

at $X=0$, $\theta=\eta$, $b_1 = B(\eta-U)^n$

at $X=1$, $\dfrac{d\theta}{dX}=0$

These boundary conditions together with Eq (16) give:

$$b_0 = \eta \tag{17a}$$

$$b_2 = -\frac{b_1}{2} \tag{17b}$$

Substitution of Eqs (16) and (17) into Eq (6) gives:

$$\frac{d}{dF}\left[b_0 + \frac{b_1}{2} + \frac{b_2}{3}\right] = -\left[\frac{\partial\theta}{\partial X}\right]_{X=0} \tag{18}$$

Substitution of Eqs (17) into Eq (18) and recalling that

$$\left[\frac{\partial\theta}{\partial X}\right]_{X=0} = -b_1$$

gives:

$$\frac{d\eta}{b_1} + \frac{1}{3}\frac{db_1}{b_1} = -dF \tag{19}$$

Integration of Eq (19) between the limits (F^*,η^*) and (F,η) gives the inverse solution for the post-penetration period:

$$B^2F = B^2F^* + \frac{B}{(n-1)}\left[\frac{1}{(\eta-U)^{n-1}} - \frac{1}{(\eta^*-U)^{n-1}}\right] + \frac{nB^2}{3}\ln\left[\frac{(\eta^*-U)}{(\eta-U)}\right] \tag{20}$$

For any chosen value of the dimensionless surface temperature the corresponding dimensionless time may again be evaluated, and the complete temperature profile estimated from Eqs (16) and (17).

Convective heating

The temperature response in a plate heated by convection may similarly be obtained but it should be noted that, because of the nonlinear nature of the problem, the solution is not symmetrical with the cooling problem. For completeness the solution to the heating problem is included here. Temperatures are now rendered dimensionless with respect to the environment temperature T_a, and the surface boundary flux condition replacing Eq (13) is:

$$-a_1(= -b_1) = B(1-\beta)^n \tag{21}$$

The resulting pre-penetration solution is:

$$B^2F = \frac{2}{3}\left[\frac{(1-\gamma)}{(2n-1)(1-\beta)^{2n-1}} - \frac{1}{(2n-2)(1-\beta)^{2n-2}} + \frac{1}{(1-U)^{2n-2}}\left[\frac{1}{2n-1} + \frac{1}{2n-2}\right]\right] - \frac{1}{2}\frac{(\gamma-\beta)^2}{(1-\beta)^{2n}} \tag{22}$$

The dimensionless surface temperature at penetration, β^*, may be estimated from the equation:

$$B(\beta^*-1)^n - 2(\gamma-\beta^*) = 0 \tag{23}$$

and the solution in the post-penetration period is given by:

$$B^2F = B^2F^* + \frac{B}{(n-1)}\left[\frac{1}{(\beta-\gamma)^{n-1}} - \frac{1}{(\beta^*-\gamma)^{n-1}}\right] + \frac{nB^2}{3}\ln\left[\frac{\beta^*-\gamma}{\beta-\gamma}\right] \tag{24}$$

where F^* is calculated from Eq (22) with $\beta = \beta^*$.

Results

For the purposes of assessing the accuracy of the integral method a comparison was first of all made between the analytical and integral solutions for the problem of a plate cooling by forced convection. The analytical solution of this problem is given by Carslaw and Jaeger[3] and may be conveniently rewritten as:

$$\eta = U + (1-U)\sum_{i=1}^{\infty} \frac{2\sin 2\lambda_i}{2\lambda_i + \sin 2\lambda_i}\exp(-\lambda_i^2 F) \qquad (25)$$

where λ_i are the successive roots of the equation

$$B\cos\lambda = \lambda \sin\lambda \qquad (26)$$

The number of terms required to obtain convergence of Eq (25), particularly when B is large, tends to infinity as $F \to 0$. Since only the first nine terms of the series were used for the purposes of comparison in this paper the results obtained using Eq (25) must be treated with caution during the starting period of the cooling process. This starting period becomes longer as B increases. The numerical solution of this problem also exhibits a similar instability.

The pre-penetration integral solution, obtained from Eqs (12) and (13) with $n=1$, is:

$$B^2F = \frac{2}{3}\left[\frac{(1-U)}{(\eta-U)} - 1 + \ln\left[\frac{\eta-U}{1-U}\right]\right] + \frac{1}{3}\left[\frac{1-\eta}{\eta-U}\right]^2 \qquad (27)$$

and the corresponding post-penetration solution is:

$$B^2F = B^2F^* + \left[B + \frac{1}{3}\right]\ln\left[\frac{\eta^* - U}{\eta - U}\right] \qquad (28)$$

which applies once the dimensionless surface temperature has fallen below $\eta^* = (BU+2)/(B+2)$.

The cooling history of a plate may be predicted using Eqs (27) and (28) with a pocket calculator. A comparison of the results obtained using Eq (25) and those obtained using Eqs (27) and (28) is given in Table 1.

Also included in Table 1 is the prediction of the cooling history for the same problem obtained using a Crank–Nicolson[4] technique. The particular form of the Crank–Nicolson method used here employed central difference approximations for time and space derivatives. Stability of the method depends upon the value of F. At high values of F Cameron[5] has shown that five space intervals are sufficient to achieve accurate results. However, at lower values of F (ie short elapsed times) the number of required space intervals increases. The results given in Table 1 were obtained using a variable computational net in such a way as to obtain reasonable agreement with the analytical solution.

The conclusions which may be drawn from the data in Table 1 are therefore that the integral analysis produces results which are accurate enough for many engineering purposes and that the Crank–Nicolson scheme may be adopted as a secondary standard for use in testing the performance of the integral analysis of the free convection problem.

Figs 1, 2 and 3 show the dimensionless surface

Table 1 Comparison of analytical, numerical and integral solutions for the cases $U=0.25$, $B=0.1$, 1 and 10 with $n=1$

		η		
B	B^2F	Analytical (Eq (25))	Integral (Eqs (27/28))	Numerical (Crank-Nicolson)
0.1	0.373×10^{-2}	0.9501	0.9500	0.9501
	0.114×10^{-1}	0.8997	0.9000	0.8998
	0.197×10^{-1}	0.8496	0.8500	0.8497
	0.286×10^{-1}	0.8001	0.8000	0.8002
	0.385×10^{-1}	0.7499	0.7500	0.7500
	0.494×10^{-1}	0.6999	0.7000	0.6999
	0.616×10^{-1}	0.6498	0.6500	0.6498
	0.754×10^{-1}	0.5998	0.6000	0.5998
	0.913×10^{-1}	0.5499	0.5500	0.5499
	0.110	0.5003	0.5000	0.5002
	0.133	0.4504	0.4500	0.4503
	0.163	0.3998	0.4000	0.3998
	0.205	0.3498	0.3500	0.3498
	0.276	0.3002	0.3000	0.3002
1	0.332×10^{-2}	0.9512	0.9500	0.9537
	0.151×10^{-1}	0.9064	0.9000	0.9064
	0.387×10^{-1}	0.8588	0.8500	0.8588
	0.797×10^{-1}	0.8102	0.8000	0.8103
	0.146	0.7611	0.7500	0.7611
	0.287	0.6964	0.7000	0.6964
	0.444	0.6447	0.6500	0.6446
	0.622	0.5955	0.6000	0.5954
	0.827	0.5468	0.5500	0.5467
	1.070	0.4979	0.5000	0.4978
	1.370	0.4486	0.4500	0.4484
	1.750	0.3999	0.4000	0.3997
	2.290	0.3505	0.3500	0.3503
	3.220	0.3005	0.3000	0.3002
10	0.252	0.7088	0.7000	0.7111
	0.419	0.6601	0.6500	0.6605
	0.689	0.6090	0.6000	0.6092
	1.140	0.5572	0.5500	0.5573
	1.930	0.5054	0.5000	0.5055
	3.470	0.4530	0.4500	0.4531
	6.930	0.4009	0.4000	0.4009
	20.100	0.3415	0.3500	0.3415
	50.200	0.2980	0.3000	0.2979

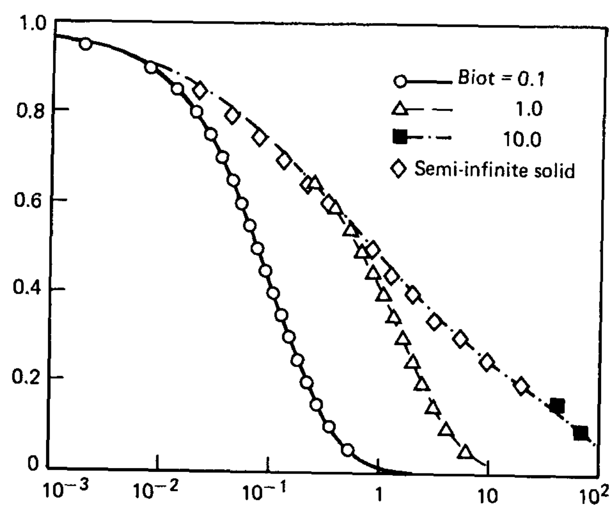

Fig 1 Dimensionless surface temperature histories of a plate cooled by natural convection, n = 4/3, U = 0, B = 0.1, 1 and 10; points from Eqs (14) and (20), lines from Crank–Nicolson solution

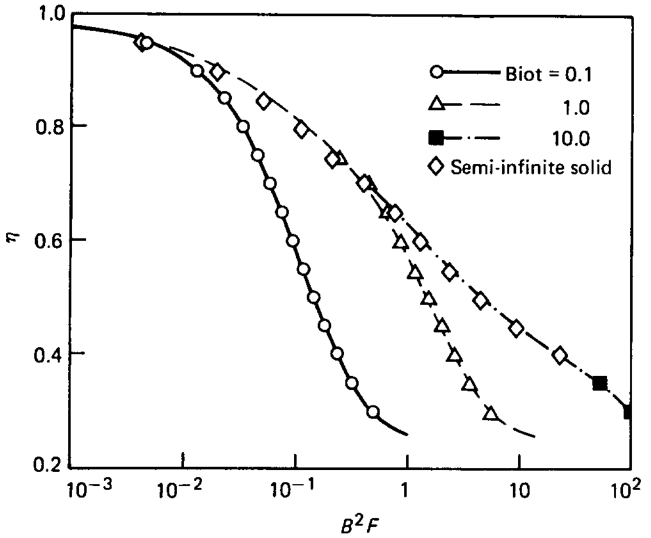

Fig 2 Dimensionless surface temperature histories of a plate cooled by natural convection, n = *4/3,* U = *0.25,* B = *0.1, 1 and 10; points from Eqs (14) and (20), lines from Crank–Nicolson solution*

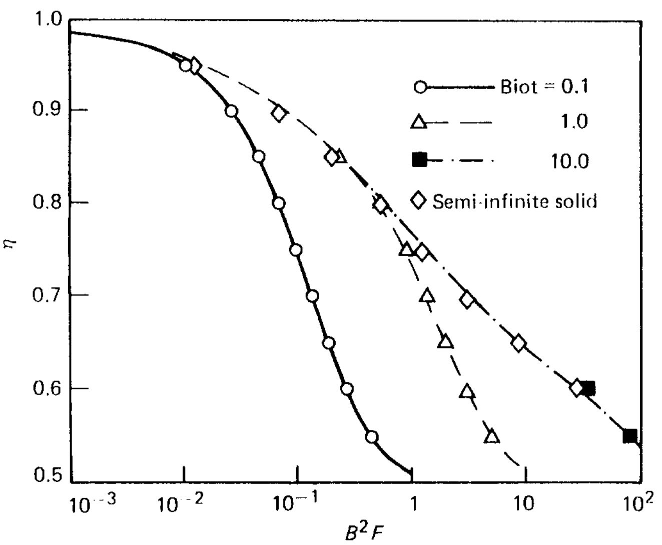

Fig 3 Dimensionless surface temperature histories of a plate cooled by natural convection, n = *4/3,* U = *0.5,* B = *0.1, 1 and 10; points from Eqs (14) and (20), lines from Crank–Nicolson solution*

temperature histories of a plate cooling by turbulent natural convection for values of $U = 0$, 0.25 and 0.5. In each case values of the Biot modulus, representing a range of possible practical situations, are 0.1, 1 and 10.

The curves in Figs 1, 2 and 3 were produced using the Crank–Nicolson scheme. Since the heat transfer coefficient was a function of temperature the variation of the coefficient was treated by using sufficient time intervals to eliminate any significant errors. The points in Figs 1, 2 and 3 were predicted using Eqs (14) and (20).

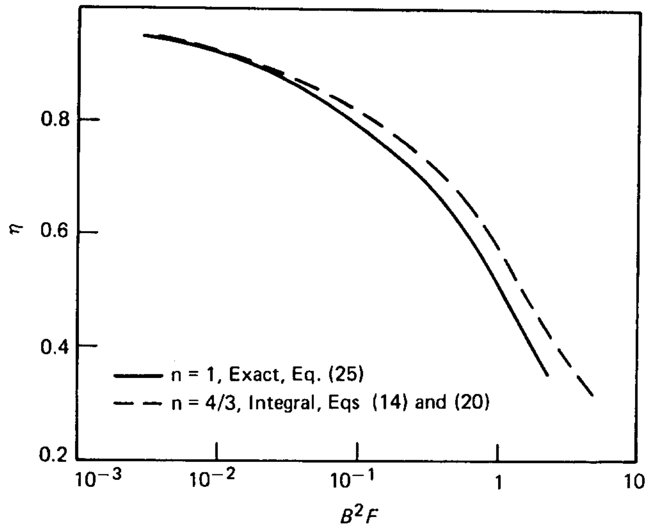

Fig 4 Dimensionless surface temperature histories of a plate cooled by forced convection (n = *1) and natural convection* (n = *4/3) for* U = *0.25,* B = *1*

Fig 4 shows a comparison of the predicted dimensionless surface temperature histories of a plate cooling by forced convection (Eq (25)) and by natural convection (Eqs (14) and (20)) for the case $U = 0.25$, $B = 1$.

Conclusions

An approximate solution of the nonlinear problem of transient conduction in a plate with a free convection boundary condition has been shown to exist in terms of elementary functions.

The related solution for the linear forced convection boundary condition is shown to agree well with the exact solution. The integral solution for the free convection boundary condition is shown to agree well with a numerical solution.

Appreciable differences can occur between the cooling histories of a plate subjected to either free or forced convection for similar values of the Biot modulus.

The utility of the integral solutions for both free and forced convection boundary conditions is such that a numerical result for any practical problem can be obtained using a pocket calculator. Alternative methods require considerable computing power to cover a similar range of F and B.

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